A2 Mathematics Unit 3: Pure Mathematics B
General instructions for marking GCE Mathematics

1. The mark scheme should be applied precisely and no departure made from it. Marks should be awarded directly as indicated and no further subdivision made.

2. Marking Abbreviations
The following may be used in marking schemes or in the marking of scripts to indicate reasons for the marks awarded.

cao = correct answer only
MR = misread
PA = premature approximation
bod = benefit of doubt
oe = or equivalent
si = seen or implied
ISW = ignore subsequent working

F.T. = follow through (√ indicates correct working following an error and ✗ indicates a further error has been made)

Anything given in brackets in the marking scheme is expected but, not required, to gain credit.

3. Premature Approximation
A candidate who approximates prematurely and then proceeds correctly to a final answer loses 1 mark as directed by the Principal Examiner.

4. Misreads
When the data of a question is misread in such a way as not to alter the aim or difficulty of a question, follow through the working and allot marks for the candidates' answers as on the scheme using the new data.
This is only applicable if a wrong value, is used consistently throughout a solution; if the correct value appears anywhere, the solution is not classed as MR (but may, of course, still earn other marks).

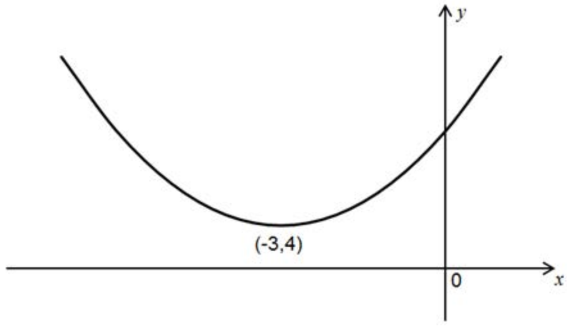
5. Marking codes

- 'M' marks are awarded for any correct method applied to appropriate working, even though a numerical error may be involved. Once earned they cannot be lost.
- 'm' marks are dependant method marks. They are only given if the relevant previous 'M' mark has been earned.
- 'A' marks are given for a numerically correct stage, for a correct result or for an answer lying within a specified range. They are only given if the relevant M/m mark has been earned either explicitly or by inference from the correct answer.
- 'B' marks are independent of method and are usually awarded for an accurate result or statement.
- 'S' marks are awarded for strategy
- 'E' marks are awarded for explanation
- 'U' marks are awarded for units
- 'P' marks are awarded for plotting points
- 'C' marks are awarded for drawing curves

A2 Mathematics Unit 3: Pure Mathematics B

Solutions and Mark Scheme

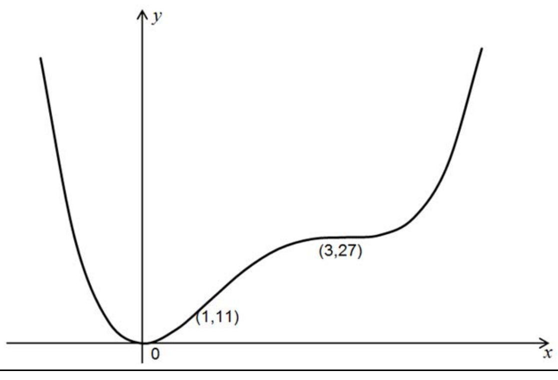
Question Number	Solution	Mark	AO	Notes
1. (a)	$1-\frac{x^2}{2}-4x=x^2$	M1	AO1	(Attempt to substitute for $\cos x, \sin x$)
	$\frac{3x^2}{2}+4x-1=0$	A1	AO1	(Correct)
	$3x^2+8x-2=0$	B1	AO1	
	$x=\frac{-8\pm\sqrt{64+24}}{6}=\frac{-8\pm\sqrt{88}}{6}$			
	$x=0.230(1385...)\ ,(-2.896805...)$	B1	AO1	
		[4]		
2.	$V=\frac{4}{3}\pi r^3$			
	$\frac{\mathrm{d}V}{\mathrm{d}t}=3\times\frac{4}{3}\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$	B1	AO3	
	$4\pi\times 15^2\frac{\mathrm{d}r}{\mathrm{d}t}=250$	M1	AO3	(Substitution of data)
	$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{250}{900\pi}\approx 0.088\,(\text{cm/second})$	A1	AO3	
		[3]		

Question Number	Solution	Mark	AO	Notes
3. (a)		G1 G1	AO1 AO1	(Shape) (Stationary point)
(b) (i)	A correct statement, eg. f^{-1} doesn't exist because f is not a one-one function	E1	AO2	
(ii)	Any appropriate domain eg. There are many possible appropriate domains. It is essential that any domain must be contained in one branch of the curve shown.	B1	AO2	
	Here we consider $(-3, \infty)$. Let $y = x^2 + 6x + 13$ $= (x+3)^2 + 4$	M1	AO1	(Attempt to find *x* in terms of *y*)
	$x + 3 = \pm\sqrt{y-4}$			
	So that $x = -3 \pm \sqrt{y-4}$	A1	AO1	
	Since $x > -3$, the positive sign is appropriate	A1	AO2	
	$\therefore x = -3 + \sqrt{y-4}$			
	And $f^{-1}(x) = -3 + \sqrt{x-4}$	A1	AO2	
		[8]		

Question Number	Solution	Mark	AO	Notes
4. (a)	$(1-x)^{-\frac{1}{2}} = 1+\frac{x}{2}+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2}+..$			
	$=1+\frac{x}{2}+\frac{3x^2}{8}+...$	B1	AO1	
	Valid for $\lvert x\rvert<1$	B1	AO1	
	When $x=\frac{1}{10}$, $\left(\frac{9}{10}\right)^{-\frac{1}{2}} \approx 1+\frac{1}{20}+\frac{3}{800}=\frac{843}{800}$	B1	AO2	
	So that $(10)^{\frac{1}{2}} = 3\text{x}\frac{843}{800}=\frac{2529}{800}$	B1	AO1	
		[4]		
5.	After 30 years, saving is			
	$(1.08)1000+(1.08)^2 1000+.......+(1.08)^{30}1000$	B1	AO3	
	This is G.P with $a=(1\cdot 08)1000$			
	$r=1\cdot 08$			
	and $n=30$	B2	AO3,AO3	(B2 for 3 correct, B1 for 2 correct)
	Then			
	$S_{30}=(1000)(1.08)\left(\frac{(1.08)^{30}-1}{0.08}\right)$	M1	AO3	(correct formula)
	$\approx £122,346$	A1	AO3	
		[5]		

Question Number	Solution	Mark	AO	Notes
6.	If smallest side is a, largest side $=8a$			
	$8a = a + 14d$	M1	AO3	(Attempt to relate the two sides)
	$a = 2d$	A1	AO3	
	Perimeter $=\frac{15}{2}[2a+14d]=\frac{15}{2}.18d=135d$	M1	AO3	
	$\therefore 135d = 270$			
	$d = 2$			
	Length of smallest side $= a = 2d = 4$ cm	B1	AO3	
	Alternative mark scheme:			
	smallest side = a, largest side $=8a$			
	Perimeter $=\frac{15}{2}[a+8a]=\frac{15}{2}.9a=\frac{135}{2}a$	(M1)	(AO3)	
		(A1)	(AO3)	
	$\therefore \frac{135}{2}a = 270$	(M1)	(AO3)	
	$a = 4$	(A1)	(AO3)	
	Length of smallest side $= a = 4$ cm	**[4]**		

Question Number	Solution	Mark	AO	Notes
7. (a)	$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 36$	M1	AO2	(attempt to find $\frac{d^2y}{dx^2}$, 2 correct terms)
	For point of inflection at $(1,11)$ $12a + 6b + 36 = 0$ So that $2a + b + 6 = 0$ (1)	A1	AO2	
(b)	Also $a + b + 18 = 11$ (2)	B1	AO1	
		M1	AO1	(Attempt to solve for a, b)
	From (1), (2), $a = 1$, $b = -8$	A1	AO1	
	$\therefore \frac{d^2y}{dx^2} = 12x^2 - 48x + 36$ $= 12(x^2 - 4x + 3) = 12(x-1)(x-3) = 0$	M1	AO2	
	$\therefore \frac{d^2y}{dx^2} = 0$ when $x = 3$	A1	AO2	
	and $\frac{d^2y}{dx^2}$ changes sign as x passes through 3	m1	AO2	
	$\therefore$ There is a point of inflection	A1	AO2	(Only if m1 is awarded)
	at $x = 3$, $y = 3^4 - 8.3^3 + 18.3^2 = 27$, i.e at $(3, 27)$	A1	AO2	
(c)	$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x = 0$	M2	AO1,AO1	(M1 for correct differentiation but not equal to 0)
	$\therefore 4x(x^2 - 6x + 9) = 0$			
	giving $x = 0, x = 3$	A1	AO1	(point of Inflection) (Two Values)
	Then at $x = 0, y = 0$ and $\frac{d^2y}{dx^2} = 36$ There is a minimum at $x = 0, y = 0$	A1	AO1	
	(graph, shown below the table)	G1	AO1	general shape
		G1	AO1	min two points of inflection
		[16]		



Question Number	Solution	Mark	AO	Notes
8 (a) (i)	$-\dfrac{e^{-3x+5}}{3}+C$	M1 A1	AO1 AO1	($ke^{-3x+5)}$) ($k=-\dfrac{1}{3}$)
(ii)	$\int x^2 \ln x \, dx$ $u=\ln x, \dfrac{dv}{dx}=x^2$ $\dfrac{du}{dx}=\dfrac{1}{x}, v=\dfrac{x^3}{3}$ $\int x^2 \ln x \, dx=\dfrac{x^3}{3}\ln x-\int\dfrac{x^3}{3}.\dfrac{1}{x}dx$ $=\dfrac{x^3}{3}\ln x-\dfrac{x^3}{9}+C$ (Penalise omission of C once only)	M1 A1,A1 A1	AO1 AO1, AO1 AO1	(Correct u and $\dfrac{dv}{dx}$)
(b)	$\int_0^{\frac{1}{2}}\dfrac{x^2}{\sqrt{1-x^2}}dx$ $x=\sin\theta \quad dx=\cos\theta\, d\theta$ $x=0, \quad \theta=0 \quad x=\dfrac{1}{2}, \quad \theta=\dfrac{\pi}{6}$ $=\int_0^{\frac{\pi}{6}}\dfrac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta$ $=\int_0^{\frac{\pi}{6}}\dfrac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta$ $=\int_0^{\frac{\pi}{6}}\sin^2\theta\, d\theta$ $=\int_0^{\frac{\pi}{6}}\dfrac{1-\cos 2\theta}{2}d\theta$ $=\left[\dfrac{\theta}{2}-\dfrac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{6}}$ $\dfrac{\pi}{12}-\dfrac{\sin\frac{\pi}{3}}{4}-0+0=\dfrac{\pi}{12}-\dfrac{\sqrt{3}}{8}$	 B1 B1 M1 A1 A1 m1 A1 A1 **[14]**	 AO3 AO3 AO3 AO3 AO3 AO3 AO3 AO3	 (attempt to substitute) (Correct) (both correct)

Question Number	Solution	Mark	AO	Notes
9.	$x^2 + 4 = 12 - x^2$ $2x^2 = 8$ $x = \pm 2$	M1 A1	AO3 AO3	(Equating *y*'s)
	Area $= \int_{-2}^{2} \{12 - x^2 - (x^2 + 4)\}\ \mathrm{d}x$ $= \int_{-2}^{2} (8 - 2x^2)\mathrm{d}x$	M1	AO3	(expressing area)
	$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$	A2	AO3 AO3	(F.T arithmetic error)
	$= \frac{64}{3}$	A1	AO3	(c.a.o)
	Alternative mark scheme for the Area:			
	Area $= \int_{-2}^{2} (12 - x^2)\mathrm{d}x - \int_{-2}^{2} (x^2 + 4)\,\mathrm{d}x$	(M1)	(AO3)	
	$= \left[12x - \frac{x^3}{3} - \frac{x^3}{3} - 4x\right]_{-2}^{2}$	(A2)	(AO3) (AO3)	(A2 for 4 terms correct, A1 for 2 terms correct)
	$= \frac{64}{3}$	(A1)	(AO3)	(c.a.o)
		[6]		

Question Number	Solution	Mark	AO	Notes
10. (a)	$f(x) = 1 + 5x - x^4$ $f(1) = 5,\ f(2) = -5$	M1	AO2	(Use of Intermediate Value Theorem.)
	There is a change of sign indicating there is a root between 1 and 2.	A1	AO2	(correct values and conclusions)
(b)	$x_{n+1} = \sqrt[4]{1+5x_n},\quad x_0 = 1\cdot5,\quad x_1 = 1.707476485$	B1	AO1	
	$x_2 = 1.75734609$	B1	AO1	
	$x_3 = 1.7687213,\quad x_4 = 1.7712854$			
	$x_5 = 1.771861948,\ \alpha \approx 1.77$	B1	AO1	
(c)	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{1+5x_n - x_n^4}{5-4x_n^3}$	M1	AO1	Attempt to use Newton-Raphson
		A1	AO1	All terms correct
	$x_0 = 1.5$			
	$x_1 = 1.904411765$	M1 A1	AO1 AO1	
	$x_2 = 1.788115338$			
	$x_3 = 1.772305156$			
	$x_4 = 1.772029085$			
	$x_5 = 1.772028972$	A1	AO1	
	Root $\alpha \approx 1.772029$	A1	AO1	Correct to 6 decimal places
		[11]		

Question Number	Solution	Mark	AO	Notes
11. (a)	$4x^3 + 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$	B2	AO1,AO1	(B2, 4 correct terms) (B1, 3 correct terms)
	Now, $x = -1, y = 3$ so that $-4 - 6 + \frac{dy}{dx} + 6\frac{dy}{dx} = 0$	B1	AO1	
	$\frac{dy}{dx} = \frac{10}{7}$	B1	AO1	
(b)	$\frac{dy}{dx} = \frac{dy}{dp} / \frac{dx}{dp} = \frac{2}{2p} = \frac{1}{p}$	M1 A1	AO1 AO1	
	Gradient of normal is $-p$	B1	AO1	
	Equation of normal is $(y - 2p) = -p(x - p^2)$	m1	AO1	
	$y - 2p = -px + p^3$			
	so that $y + px = 2p + p^3$	A1	AO1	convincing
	When $y = 0,\quad x = b$ $b = 2 + p^2$	B1	AO2	
	Since $p^2 > 0,\ b > 2$	E1	AO2	
		[11]		

Question Number	Solution	Mark	AO	Notes
12. (a)	Let $y = \cos x$			
	$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{\cos(x+h) - \cos x}{h}\right]$	M1	AO2	
	$= \lim_{h \to 0}\left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}\right]$	A1	AO2	
	As h approaches 0 $\cos h \approx 1 - \frac{h^2}{2}$ and $\sin h \approx h$			
	So $\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{\cos x\left(1 - \frac{h^2}{2}\right) - \sin x \times h - \cos x}{h}\right]$	M1	AO2	
	$= \lim_{h \to 0}\left[\frac{-\frac{h^2}{2}\cos x - h\sin x}{h}\right]$	A1	AO2	
	$= -\sin x$	A1	AO2	
(b) (i)	$\frac{(x^3+1)6x - 3x^2(3x^2)}{(x^3+1)^2}$	M1	AO1	(Correct formula)
	$= \frac{3x(2-x^3)}{(x^3+1)^2}$	A1	AO1	
(ii)	$3x^2 \tan 3x + 3x^3 \sec^2 3x$	M1	AO1	(Correct formula)
	$= 3x^2(\tan 3x + x\sec^2 3x)$	A1	AO1	(All Correct)
		[9]		

Question Number	Solution	Mark	AO	Notes
13. (a)	$\operatorname{cosec}^2 x + \cot^2 x = 5$			
	$1 + 2\cot^2 x = 5$	M1	AO1	(Attempt to write in terms of one function)
	$\cot^2 x = 2$	A1	AO1	
	$\tan x = \pm\frac{1}{\sqrt{2}}$	A1	AO1	
	$x = 35.3,\ 215.3^\circ, 144.7^\circ,\ 324.7^\circ$	B1,B1	AO1 AO1	(each pair)
(b) (i)	$4\sin\theta + 3\cos\theta \equiv R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$			
	$R\cos\alpha = 4$ $R\sin\alpha = 3$	B1 B1	AO1 AO1	
	$R = \sqrt{3^2 + 4^2} = 5$	B1	AO1	
	$\tan\alpha = \frac{3}{4},\quad \alpha = 36.87^\circ$	B1	AO1	
	$4\sin\theta + 3\cos\theta \equiv 5\sin(\theta + 36.87^\circ)$			
(ii)	$5\sin(\theta + 36.87^\circ) = 2$			
	$\sin(\theta + 36.87^\circ) = 0.4$	B1	AO1	
	$\theta + 36.87^\circ = 23.58^\circ, 156.42^\circ, 383.58^\circ$			
	$\theta = 119.5(5)^\circ,\ 346.7(1)^\circ$ $= 120^\circ, 347^\circ$ to the nearest degree	B1 B1	AO1 AO1	
		[12]		

Question Number	Solution	Mark	AO	Notes
14. (a)	$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\frac{\mathrm{d}h}{\mathrm{d}t}$			
	$4\frac{\mathrm{d}h}{\mathrm{d}t} = 0.004 - 0.0008h$	M1	AO3	(3 terms, at least 2 correct)
	$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.001 - 0.0002h$			
	$5000\frac{\mathrm{d}h}{\mathrm{d}t} = 5 - h$	A1	AO3	(Correct)
(b)	$5000\int \frac{\mathrm{d}h}{5-h} = \int \mathrm{d}t$	M1	AO1	(Separation of variables)
	$-5000\ \ln(5-h) = t + C$ (1)	A1,A1	AO1 AO1	(-1 if C omitted)
	$h = 0$ at $t = 0$			
	$\therefore -5000\ln(5) = C$	m1	AO1	
	Substitute in (1)			
	$-5000\ \ln(5-h) = t - 5000\ln(5)$			
	$t = 5000\ln\left(\frac{5}{5-h}\right)$	A1	AO1	
	$\therefore \left(\frac{5}{5-h}\right) = \mathrm{e}^{\frac{t}{5000}}$	M1	AO1	(Attempt to invert)
	$5 - h = 5\mathrm{e}^{\frac{-t}{5000}}$			
	$h = 5 - 5\mathrm{e}^{\frac{-t}{5000}}$	A1	AO1	
(c)	$h = 5 - 5\mathrm{e}^{\frac{-3600}{5000}}$			
	$= 2.57\,\mathrm{m}$	B1	AO1	
		[10]		

Question Number	Solution	Mark	AO	Notes
15.	$4x^2 + 9 < 12x$ $4x^2 - 12x + 9 < 0$ $(2x-3)^2 < 0$ Impossible when x is real. Contradiction so that assumption is false. $\therefore 4x + \frac{9}{x} \geq 12$	M1 A1 A1 **[3]**	AO2 AO2 AO2	(Clear fractions)